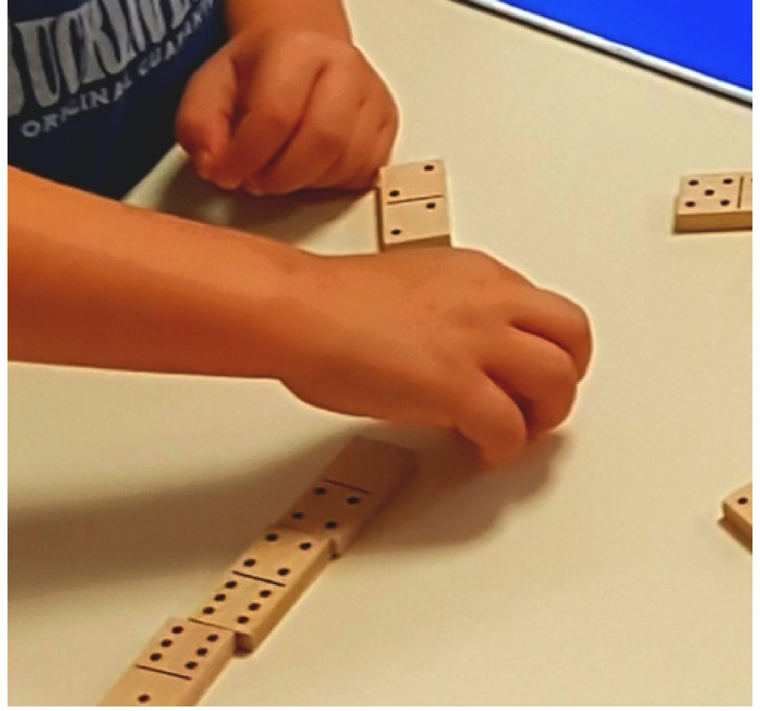
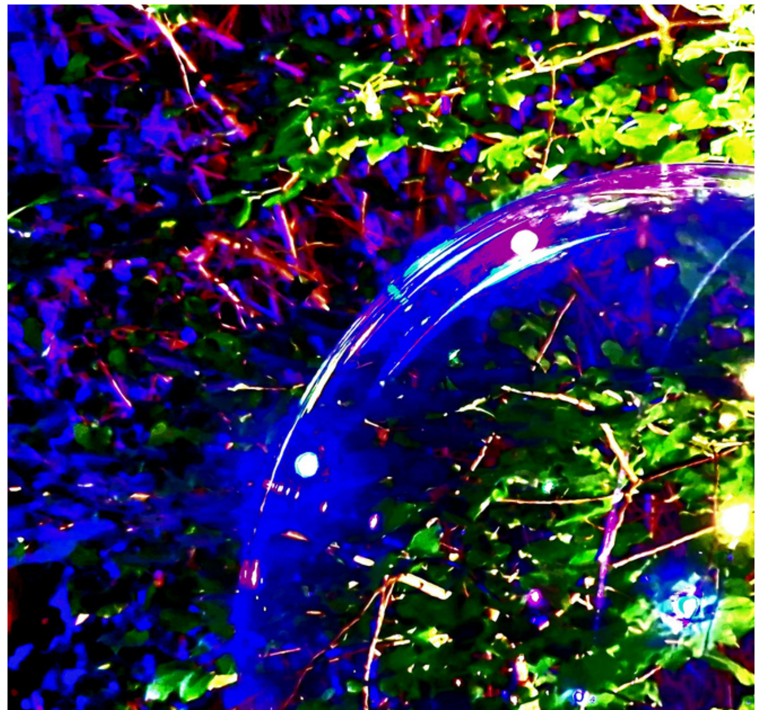
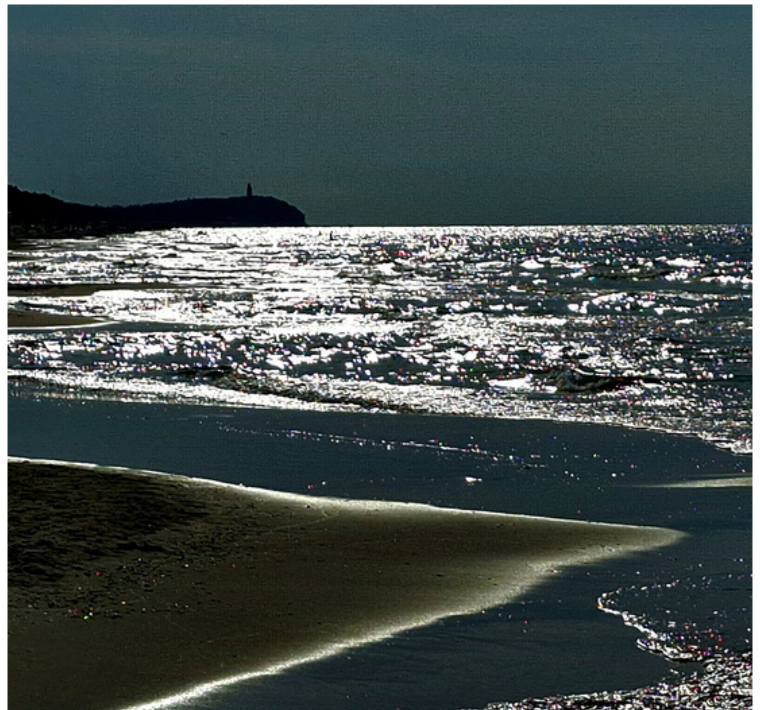




Math&Art.
Look, it's mathematics.

Małgorzata Makiewicz


Math & Art. Look, it's math.
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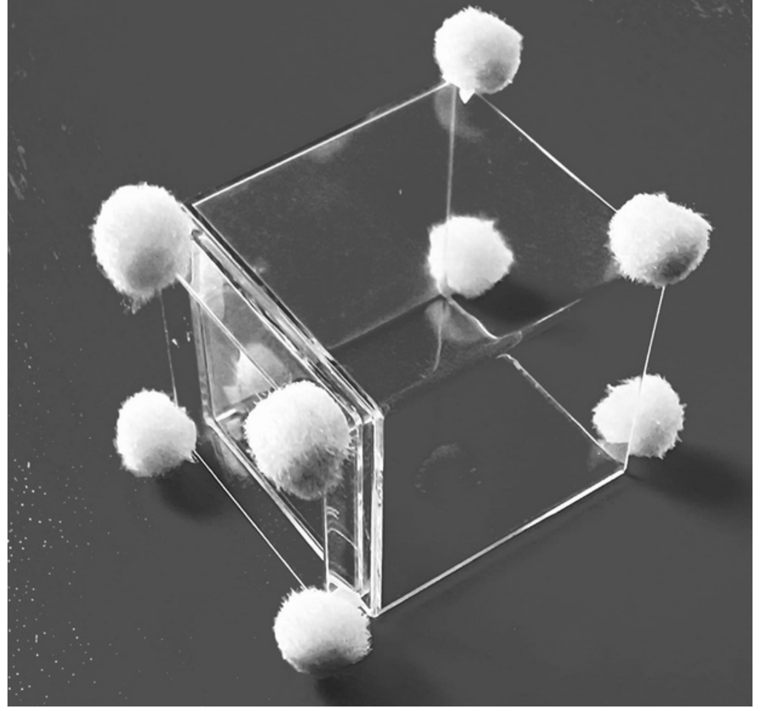
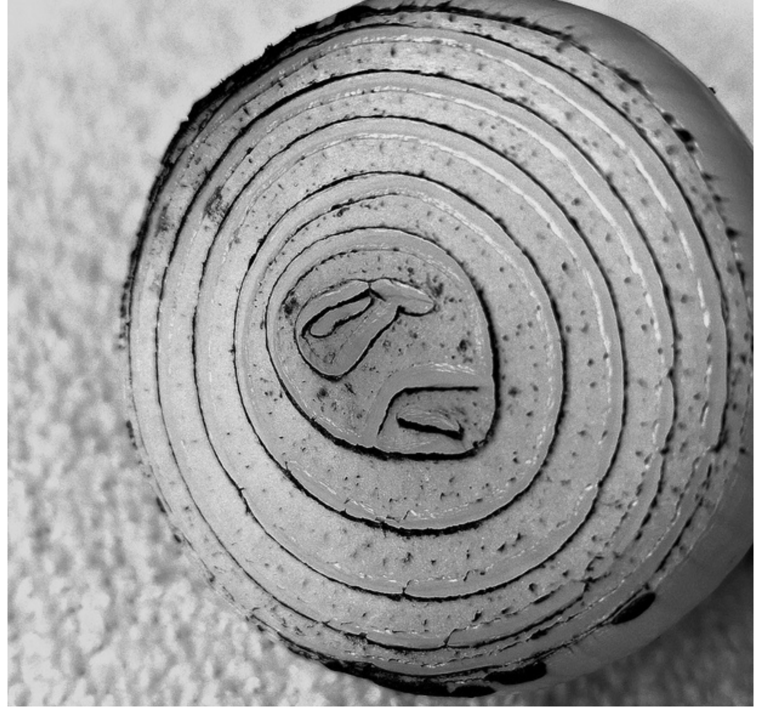

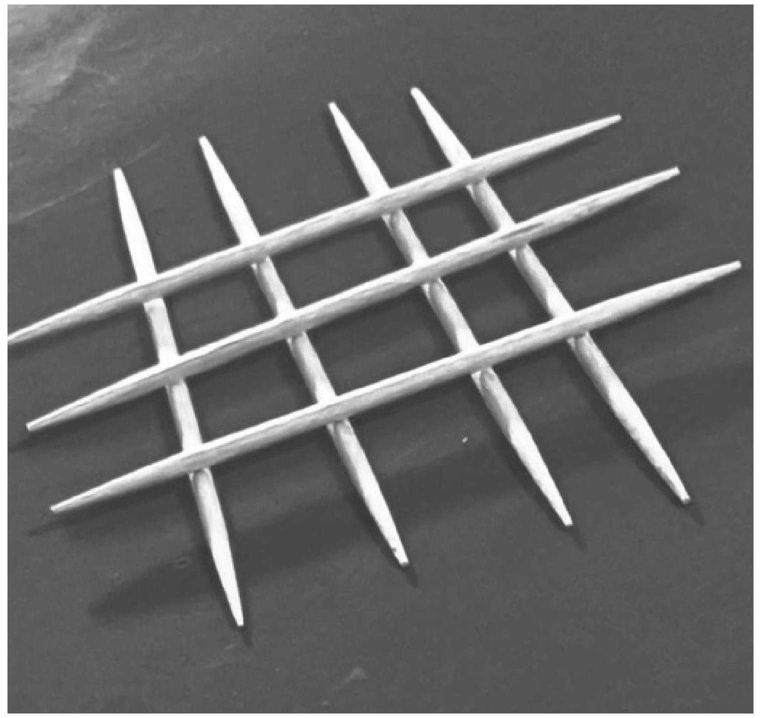


Contents

PART I (for teachers, educators, parents and guardians of preschool children)

Introduction

This book is another publication in the "Math&Art." series, prepared in the convention of visual literacy. The main way to interest a child in mathematics is joint fun with an adult and other children aimed at arousing interest and a willingness to count, measure, weigh, recognize figures, or reason. Previous books in the series were mainly addressed to educators studying educational reality and to primary and secondary school teachers. This time I turn to teachers, educators, parents, and grandparents of preschool children. The electronic edition of the book has enabled the expansion of recipients to include small viewers - preschoolers who cannot yet read on their own. The part intended for them can be "read" together with the child. The adult's task is to organize the educational space based on the ideas outlined on the pages in such a way as to enable the child to understand the essence of counting, calculating, recognizing geometric figures, or developing spatial orientation. This part of the book has been enriched with educational photos and audio recordings, which can be listened to in Polish, English, or Spanish.

The proposals included in the publication are intended to be located in the so-called zone of the child's nearest development, which, according to Vygotsky, I understand as the distance between the current and potential level of child development. The current level is the result of completed developmental cycles. The potential level is determined by the emerging mental functions influenced by teaching, imitation, other social contacts, and forms of own activity [1].

The adult's task is to support the individual development of the child, among others, by adapting the proposals of games and tasks to the child's potential capabilities. Therefore, instead of guidelines and educational recommendations, I present selected proposals whose application can contribute to a good start for the child in the field of mathematics.
The photos included in the book are intended to provoke the child to notice mathematical concepts and regularities. They also provide pedagogical guidance on how to organize the educational space of a small child in their own home, without expensive teaching aids, games, or toys. Shaping the basics of mathematical concepts and basic mathematical skills can be a wonderful adventure. A small child learns through play. Causing astonishment, curiosity, and provoking the preschooler to ask questions and formulate assumptions invites the child to explore mathematical objects.
The games proposed in the book concerning the recognition of flat figures that are shadows or stamps (stamps) of spatial models bear the hallmarks of exploratory activities. The child perceives completely new images resulting from their own manipulation of known objects (e.g., a kitchen sponge, carrot, or onion). Such activities, involving new stimuli in new situations, lead to satisfying the child's cognitive needs [2]. Meanwhile, games that engage children's ingenuity in creating new arrangements, constructions, or images made from self-made stamps play a particularly important role in the development of children's spatial imagination [3]. I encourage caregivers and educators of preschool children to creatively use gifts of nature (e.g., chestnuts, acorns, or sea shells), vegetables during meal preparation, buttons, beads, a home shadow theater. The best cognitive toy you may have at your fingertips: kitchen utensils, which you can use to teach sorting, dishes arranged in the right order, or shoes and gloves to recognize the left and right sides. Children from 3 to 7 years of age initiate constructive games, which consist of creating various constructions from elements such as blocks, pebbles, sticks, sand [4]. The thing is to skillfully direct this game so that it becomes the basis for shaping proper intuitions and outlines of mathematical concepts."

[1] „*Math & Art. Reprezentacje enaktywne w edukacji matematycznej – badania w działaniu* ”/ „*Math & Art. Enactive Representations in Math Education – Research in Action* “ i „*Math & Art. Wizualne drogi do reprezentacji symbolicznych pojęć matematycznych / Math & Art: Visual ways to represent symbolic mathematical concepts*”.

[2] L. S. Wygotski, *Wybrane prace psychologiczne*, przeł. E. i J. Flesznerowie, PWN, Warszawa 1971, s.541

[3] M. Kielar-Turska, *Jak stymulować badawczą aktywność dziecka*, „Wychowanie w Przedszkolu”, nr 9/1987, s. 503.

[4] S. Szuman, *Psychologia wychowawcza wieku dziecięcego. Podręcznik dla matek i wychowawczyń w przedszkolach i domach dziecięcych oraz dla nauczycieli i studentów studiujących rozwój psychiczny dziecka*, Instytut Wydawniczy Nasza Księgarnia, Warszawa 1946, s. 83.

[5] M. Tyszkowa, *Aktywność i działalność dzieci i młodzieży*, Warszawa 1997, s. 68.

Preschooler Counting

Number. An abstract object of mathematics that accompanies us throughout our entire life. It describes biological development, physical transformations, chemical reactions, as well as social changes, measures of length, mass, volume, speed, and thousands of other quantities. Thanks to numbers, whose understanding is assumed to be natural, we can count elements of sets (e.g., I have two legs, there are five apples in the basket, the car has four wheels), and also determine order (e.g., I live on the second floor, I took third place in the race, in the theater I have to take the eighth seat in the second row). The ability to count makes these examples understandable and practically meaningful.

In this book, I do not strictly adhere to the core curriculum of preschool education in Poland. Firstly, because I hope that the publication will reach children in other countries, and secondly, I don't think that administratively/formally limiting or delaying the range of mathematics learning for children is appropriate. Preschoolers usually have no trouble understanding negative numbers: "my dad parked the car in the shopping center parking lot on level minus two", "the verbal information in the elevator 'level zero' means we are on the ground floor" or "today the temperature dropped to minus four degrees, you need to wear a hat and gloves because it's cold". They eagerly pronounce the names of large numbers, for example, counting 680 steps during a mountain hike to Szczeliniec (a peak in the Stołowe Mountains, Sudetes). They cross the thresholds of tens and hundreds, unknowingly learning the structure of the positional counting system. Note that, unlike additive systems (like the Roman system), understanding the positional method of writing numbers requires understanding the dual role of zero: as a number and a sign indicating the absence of elements of a given order.

The ability of a preschooler to count fluently forms the basis for developing their ability to perform calculations (operations), to organize, to solve tasks based on numerical dependencies and regularities. A good understanding of numbers and numerals gives the child a basis for coding and encrypting information.

Numbers mark houses and buildings, every person is assigned a digital code - a PESEL number, and parents withdraw money from ATMs after reading the card and entering a security code - a digital pin. In addition, fluent counting without limiting the scope gives a chance to independently notice regularities occurring in positional counting systems. It also makes the child feel a sense of harmony and a certain kind of order, rhythm. Mastering the skill of counting is based on understanding and applying certain principles published in the 1970s. They are general enough to allow counting all objects in the same way. Each number is assigned a name unequivocally. Besides, the order of counted objects does not matter, but the order of numerals is fixed. The numeral that appears last during counting expresses the number of elements in the set [1].

Applying these principles in children's play leading to learning to count, we count elements of a set arranged one after another, grouped in pairs, threes, tens, or we do not change the arrangement of elements at all, but count by moving our finger from one item to another. At the same time, we pronounce successive numerals. It turns out (and this is a children's discovery) that regardless of the position, the number of elements in the set is the same.

Introducing a child to logical reasoning and expressing their thoughts takes years, based on daily stimulation. This involves both deliberate and planned provocation (e.g., during preschool education) and spontaneous ones, which occur, for example, during a walk with a grandparent visits to the cinema or evening fun with shadow theater before bedtime are essential, as education for the youngest students should skillfully intertwine learning with play, to gently introduce them to the world of school [2].

Encouraging a child to count correctly is facilitated through conversations during daily activities - for example, “for dinner, we will have two guests today, so we are preparing two additional settings for: grandma, grandpa, mom, dad, child,” or when sorting clothes after laundry, “we pair socks and put them in the drawer.” Home mathematical education is an irreplaceable complement to programmed preschool education, positively shaping peer relations and between children and adults. It also makes children naturally gravitate towards learning when they feel the practical application. Mathematical regularities organize our entire life, accompanying all tasks, work, learning, entertainment, and even emotions and feelings. Therefore, we should use all opportunities that arise in our life to gradually shape correct mathematical intuitions in a child.

In the epilogue to Bruno de Finetti's book, Andrzej Góralski emphasized that pedagogical action requires a balance of intellect and intuition. Hence, the challenge is for educators to master the craft of developing intuition to the same degree as the skill of educating the intellect [3].

This long-standing postulate is still relevant, especially addressed to all who contribute to the development of a child's mathematical thinking. Implementing the shaping of correct intuitions and so-called germs of mathematical concepts, and thereby achieving a child's readiness to learn mathematics, may reduce the blocks in mathematical education in young children, as described by Edyta Gruszczyk-Kolczyńska [4]. It's worth noting that excessive difficulties in learning mathematics appear at the beginning of school education, becoming clearly evident in the fourth grades of primary schools. Accumulating backlogs replace the initial fascination with school and gradually turn into dissatisfaction, disappointments, frustrations, and a long-entrenched aversion to mathematics. The consequence of such negative events is the systematic extinguishing of interests and abilities and the accumulation of cognitive backlogs during the entire teaching cycle [5].

The mathematical cognitive adventure of a small child often begins with naming numbers, but, for over 70 years, it has been known that even infants, on the basis of subitization, i.e., the sudden, non-arithmetical perception of differences in the numerosity of sets with a small number of elements, distinguish between sets of 2 and 3 elements [6]. The process of conscious and correct counting develops through a series of preparatory activities, including, among others, estimation. A small child first captures the regularities of rhythm, using the pointing gesture.

These are the gestures that Edyta Gruszczyk-Kolczyńska says herald the possibility of counting. Pointing at an object by a walking child independently exploring objects in their environment is accompanied by establishing relationships between existing objects and their mental representation, on the basis of: one pointing gesture - one real object - one representation in the child's mind. This is the beginning of the “one-to-one” rule used in counting [7]. The path to correct counting goes through the stage of chaotic repetition by the child (around the fourth year of life) of random numerals two, seven, four, eight… Then comes the understanding of the fixed order of numerals and assigning them to individual elements of the counted set. I emphasize here that, despite the results published in 1999 by Brian Butterworth on the achievement of individual arithmetic skills by a child up to 7 years of age [8], these data are only indicative. They do not constitute developmental norms [9]. This is confirmed by Edyta Gruszczyk-Kolczyńska's research on the varied level of object counting skills in preschool children [10]. Therefore, the second part of the book, dedicated to the youngest readers, distances itself from defining specific educational thresholds that a child of 4, 5, or 6 years “should” achieve. The book, prepared in the convention of visual literacy [11], aims to guide the young recipient towards conscious viewing and reasoning, internalizing perceived regularities through the extraction and interpretation of photographic images. The goal of the book oscillates between awakening a child's aesthetic sensitivity, focused on transitioning from passive seeing to active looking [12], and shaping mathematical thinking, the essence of which is rational perception, reasoning, analyzing processes, formulating and solving problems [13]. My intention is also to contribute, albeit modestly, to building the contemporary language of mathematical communication for small children.

My long-standing teaching practice, as well as my experience in training future mathematics teachers at all educational levels, encourages me to pay attention to common inaccuracies, and even errors, in the use of basic terminology related to numbers and counting. As I have already mentioned, a number is an abstract object, difficult for a child at the preoperational stage of intellectual development to understand. Defining a natural number as a mental unit of many units [14] refers to counting the elements of a set by adding another element. The understanding of a natural number as the power (number of elements in a finite set) originating from Bertrand Russell and Friedrich Ludwig Gottlob Frege leads to assigning numerals to classes of sets with the same cardinality. This means that regardless of the elements of the set, we assign them a "label" indicating the number of these elements (five grains of sand, five apples, five horses, five houses, etc.). At the preoperational stage, understanding that a set of five grains and a set of five houses have the same number of elements is challenging because the size and extent of the elements speak louder to the child than their number. Therefore, during counting lessons, we consider various sets of things close to the child. When we are at home, these can be: cutlery, dishes, clothes, books, toys, flowers on the windows, shoes, pictures; during a trip: cars, bicycles, road posts, lamps, stairs; while gardening: beds, fruits, vegetables; during play: cars, dolls, teddy bears.

A special role in shaping a child's way of counting is played by counting on fingers. This is the first and always with us, usually not hidden under clothing, didactic tool used both to determine the number of elements in a set (e.g., three books) and to give an ordinal number indicating the order of objects (e.g., page five). We use fingers to point, touch, and move individual elements of the counted set. They are also applicable in performing basic operations on natural numbers. Monika Szczygieł, Krzysztof Cipora, and Mateusz Hohol describe research findings showing a good correlation between children's proficiency in using fingers, naming them, and numbering with their high mathematical abilities [15].

In the case of sets with a small number of elements, a child, even without the help of fingers or other substitute sets, counts these elements in both sets. If they obtain equal numbers, it means that the sets have the same cardinality. For example, two shoes and two legs or four plates for four family members. It turns out that a preschooler, even if his counting range is (still) small, can perfectly check the cardinality of two, even very numerous, sets. He establishes a correspondence between the elements of one set and the elements of the second set. Such an exercise can be done in the park during an autumn walk: "let's check if we have the same number of chestnuts and acorns," at tea time: "do we have the same number of cookies and plates?" During a seaside rest, pebbles, shells on the beach invite you to play with cardinality. How to check if we have the same number of shells as small pebbles? Just put one pebble in each shell. If a shell remains empty, it means there were more shells than pebbles.

The discussed issue, related to determining the cardinality of a set, is presented in the literature as the "cardinal aspect of the natural number." Since for many of my students this word was not fully understandable and was associated with the term "cardinal error," I hasten with a simple explanation: the cardinal number of a finite set is simply the number of its elements, so the cardinal aspect concerns the number of elements in a set.

Both public statements by well-known people and conversations with students, and even preschool teachers, authorize me to speak out about two concepts: digits and numbers. Do not use incorrect terms like: "I will add digit 5 to digit 7," or "the attendance of class IIIa is expressed by the digit 89%". Just as we write words in the native language (and other languages too) using letters, so we write numbers using digits, i.e., graphic signs. In the decimal positional system, any number can be written using ten digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, and in the binary system – using two digits: 0 or 1.

The same digits, within one counting system (let's take 3,4 and 7 and the decimal system) arranged in the right order mean other numbers e.g. 347, 374, 437, 473, 743,734. Each of them corresponds to a different value. Note also that the same numerical notation in different systems may mean a different value. For example, 1101 in the decimal system means the number "one thousand one hundred and one," while interpreting the same notation in the binary system, we have the number "thirteen," because 1+0+4+8=13.

The process of child counting is shaped in conjunction with learning to use language. Learning to read and write, like learning to count, involves encoding and decoding information. Mathematical language is considered one of the most important components of a student's mathematical culture [16]. Alexandre V. Borovik and Tony Gardiner [17] emphasize the lasting relationship between acquiring basic mathematical skills and educating in mathematical language. This thread will accompany considerations on shaping a child's spatial orientation and understanding of flat and spatial figures included in the book.

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Spatial Orientation in Preschool Children

In mathematical education, a good spatial imagination plays a crucial role. Its proper development not only determines success in learning mathematics but also the ability to navigate unfamiliar terrain, communicate effectively, and the capacity to perceive problems, ask questions, and[1] seek answers. According to Théodule Ribot, imagination is a process of gradually creating images that are equivalents of movement [1]. Barbara Rabijewska attributes to these images the role of memory pictures stored in consciousness independent of the direct presence of objects [2]. Antoni Pardała understands spatial imagination as the dynamic ability to actively create images reflecting spatial, flat, and linear forms, and the ability to transform these images through mental operations [3].

Stimulating and developing children's imaginations is achieved through toys, educational tools, voice modulation, and presenting images. These means aim to evoke children's activity as a result of equilibrium disruption and cognitive conflict [4]. Imagination is a process without which there can be no proper perception and naming of spatial figures, determining relationships and connections in these figures, designing, or solving spatial optimization tasks. Therefore, it is necessary at all levels of mathematical concept understanding described by Pierre van Hiele [5]: the visual level, where the student sees figures holistically, distinguishing them in terms of shape, position, the descriptive level, where the student analyzes the figures, generalizes by assigning names, discovers their properties, orders them, and the logical level, which includes ordering the properties of figures and designing.

These levels fully correspond to the stages of intellectual development described by Jean Piaget. In particular, the visual level corresponds to the pre-operational stage. In this stage, lasting from the end of the second year of life to the end of the preschool period [6], a child can refer to past experiences, talk about them, imitate them, and understand the world through action. However, concrete activities are irreversible at this time; children cannot mentally return to the original situation through reverse action (...) A characteristic feature of this period is intuitive thinking, and reasoning is based mainly on perception, which in turn takes into account only one aspect of a situation or event, disregarding other remaining aspects [7].

Well-developed spatial orientation in the pre-operational period is the basis for the development of a student's spatial imagination. It also plays a special role from the perspective of further development of mathematical skills, facilitating the understanding of concepts and learning in other areas of mathematics, beyond arithmetic [8]. Therefore, pedagogical actions supporting it are so important. Preschool children are characterized by so-called egocentrism, which includes, among other things, difficulty in seeing and taking into account the point of view of another person [9]. Preschool children distinguish terms of position change: "up, down, forward, backward" better than indicating the left and right sides [10]. Since understanding the position of other objects poses serious difficulty for preschoolers, to overcome it, I suggest first consolidating the structure of one's own body and the position of other objects relative to oneself.

This is the first step towards developing correct spatial orientation in a child, preceding the differentiation of geometric figures, understanding measures, and the ability to perform transformations [11]. Learning about one's body structure is accompanied by games, songs, rhymes, and dances, in which hands, legs, head, knees, heels, palms, ears, mouths, noses are shown. The child links recognizing parts of their body with indicating, stating functions, and correct naming and counting - "I have two hands, two legs, five fingers on each hand." A good understanding of directions (front, back, side) becomes complicated when we need to indicate the right or left side. Recognizing and naming one's left and right sides are supported by color markings - colored socks (e.g., red on the right foot, blue on the left).

Some adults, while teaching a child to recognize their left and right sides, attach a sign to one side. It could be a bow or a sticker. For allergic and aesthetic reasons, I do not recommend marking a child's hand with markers, paints, or pens. An interesting, though not easy, task is to recognize the right or left hand, foot, glove, shoe, imprint in the snow, the trace left by touching a hand on a steamed mirror. A slightly more difficult task is recognizing the right and left hands of another person "by touch," with closed eyes. Touch is an important sense, its role is recognized by educators who suggest that children practice naming known spatial figures found in an opaque bag.
A good exercise is also a drill loved by children consisting of turning or moving in a specified direction commanded, e.g., "right turn," "left turn," "two steps forward, march." Giving commands like "raise your left leg," "touch your right ear," "hop on your right foot," similar to drill commands, works great if alternately given by an adult and a child. After such exercises, the child easily develops their reference point.

Experiences with a mirror, reflection in a lake or puddle, or observing peers standing next to and opposite each other arouse children's anxiety. This is an extremely important teaching moment. Uncertainty expressed by the statement: "When Jacek stands next to me, our right feet are on the same side. Right. But when Jacek stands opposite me, his right foot is on my left side" is a step towards a better understanding of the concept. A child's mastery of recognizing the right and left sides of a person standing opposite is an intermediate stage relative to the application of any reference point. Well car mats with toy cars, blocks, or models depicting obstacles such as mountains, colorful houses, gardens, or farms prove useful in conducting Jean Piaget's study on the ability of free change of reference point. The child's responses to questions like 'what does the doll see from your right, left, or opposite you,' as described and commented on by Margaret Donaldson, reveal the child's difficulty in realizing that what they see is relative to their position [12].

Bronisław Rocławski particularly highlights the importance of good mathematics learning in preschoolers with the development of correct native language. Good understanding of the meaning of spoken words strengthens the understanding of concepts, allows two-way communication, and helps express one's ideas. "In the realm of natural language, a specific play between logic and intuition, analysis and construction, generalization and individualization can be observed. This reflects the beauty of a world that is neither entirely logical nor entirely intuitive. (...) Every mathematician will admit that there is no mathematics without language" [13]. Correct understanding of the meaning of prepositions, which are invariable parts of speech, is an important condition for good communication and proper development of spatial orientation. It turns out that even young children handle this well. Exercises like "say where the bear is" work well. The child, looking at a scene/picture/photography, responds: "the bear is in the box", "the bear is under the box", "the bear is in front of the box", "the bear is behind the box", "the bear is above the box", "the bear is beside the box".

It is good to combine the operation of recognizing and naming prepositions with a game of placing the bear according to instructions: "put the bear in front of the box", "put the bear behind the box", "put the bear under the box", "put the bear on the box", "put the bear into the box", etc. This is an activity aimed at awakening the basic mathematical activities of the student formulated by Kazimierz Skurzyński [14]. It is beneficial if we prepare a story for this game, for example, we want to pack a present - a bear in a box. Note that determining the toy's position relative to the box is not unambiguous. The statement "the bear is beside the box" fits both to the left and right sides. Thanks to prepositions, we can easily determine the position of something relative to others – for example, "the dog sits under the table", "a book lies on the desk", "a doormat lies in front of the door", "behind the house is a trash can", "under the nose are the lips", "I have a watch on my hand", "a lamp hangs above the head". The preposition "between" defines the position of something relative to two other things – for example, "between the red and blue rings of the toy centipede is a yellow ring", "between a person's eyes is the nose" or "between the knife and fork lies a plate". Only after mastering the understanding of the position of visible objects will the child start using prepositions when defining the rhythmic organization of time "between Tuesday and Thursday is Wednesday" or "between February and April is March". When learning to count, a preposition referring to the placement of a number on the number line appears "between six and eight is seven".

During the formation of skills to understand the position of things relative to others, the attractiveness of the examples used is important. They should relate to everyday situations that the child encounters at home, in preschool, on a walk, or in a shop.

For example, the following messages spoken in a supermarket become clear and understandable for the child: "cat crisps are in the second aisle on the left, behind the cleaning products", "opposite the bread shelf are the jam shelves" or "ice creams are in the cooler, to the left of the frozen fruits”

Here is the English translation of the text, designed for adults, teachers, and parents of school children: "During the joint 'reading' of graphic materials (illustrations, drawings, posters, graphics), the child practices correctly understanding the position of individual elements in the picture. The child describes: 'In the middle, there is a house, above it, a bird is flying, and beside it, a tree grows. There are two squirrels in the tree. In the upper left corner, you can see the sun, and to the left of the house, there is a kennel. A dog is sitting in the kennel.' The next stage is drawing, painting, or building with blocks based on verbal instructions. For example: 'Draw a Christmas tree consisting of three green triangles and a rectangular trunk. On the lower triangle, mark three red circles, on the middle one – three purple circles, and on the upper one – three blue circles. Draw a yellow star at the top of the tree.

Under the tree, on the right side, draw a pink box with a purple bow.' To complete this task, the child must organize space on a rectangular sheet of paper. It is important to speak the entire instruction slowly at first, and then repeat it slowly, giving the child enough time to act. Small children often start drawing from the edge of the sheet. They need to plan space for the star (above the upper triangle) and space to draw the box – under the bottom. Instead of painting or drawing with crayons, one can first use ready-made elements cut out of colored paper, stamps, or self-adhesive stickers. It is also possible to use a touchscreen or interactive whiteboard.

Developing good spatial orientation is great fun. Both for the child and for the creative educator and guardian. Searching for 'treasure' on a grid paper, playing hide-and-seek with a teddy bear in a box, or describing a doll's clothing will be attractive and educational for the child.

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Introduction to Preschooler Geometry. Solid and Plane Figures.

> The most important difference between number and space is that space seems to us something less detached and less basic than number.
>
> *Alfred North Whitehead*

Geometry. One of the seven liberal arts. As Plato said, "it draws the soul towards truth and creates a way of thinking worthy of a philosopher because it lifts our gaze from the lowlands to higher objects" [1]. Geometry was born from the problems of measuring the earth. The profession of "geometer" refers to a person measuring land and preparing maps and plans. A geometer could, for example, use a string with evenly tied knots to mark out a rectangular plot of land. To capture a right angle, he used the converse of the Pythagorean theorem and the model of an Egyptian triangle.

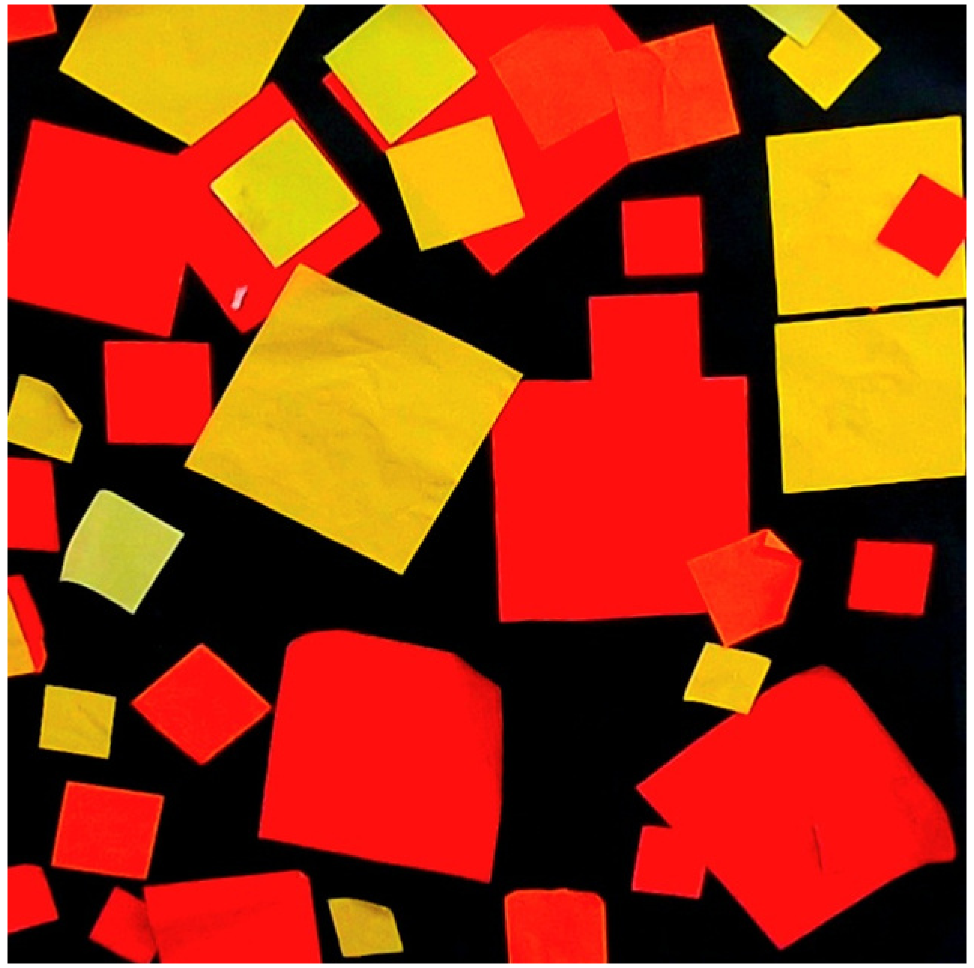
Modern geometry deals with problems connecting the understanding of plane and solid figures with algebra, topology, combinatorics. Geometry is particularly important, from the point of view of training the imagination, sensing shape, and form, as a field of school mathematics. Surprisingly, it is liked by children, although not necessarily by teachers. Teaching geometry to preschool children primarily involves: capturing the shape of a figure on a visual level (it's about recognizing and naming objects – e.g., spherical ones), identifying the properties of the figure on a descriptive level (e.g., a cube has the same sides, there are six of them, they are square), and logically examining the relationships between properties (e.g., by how much will the surface area of a cubic die increase if we attach a second one to its side?).

Experimental studies conducted by Urszula Szuścik have shown that in preschool and younger school-aged children, visual perception can be shaped, and their artistic creativity stimulated. These studies [2] laid the foundation for creating a new pedagogical concept, stimulating children's artistic activities, but also for provoking discoveries in elementary geometry through art, based on so-called "visual qualities" (color, line, and volume).

Geometry plays an essential role in developing elementary mathematical culture [3] in a child, expressed in developing curiosity and interests, noticing regularities, beauty, elegance, and harmony of basic mathematical concepts [4]. For many years, we have known that "Gothic cathedrals and Doric temples are mathematics transformed into stone" [5]. Understanding geometry helps to notice regularities in architecture, ornamentation, and social processes. It is also an excellent "material" for educating mathematical language for communication in science [6].

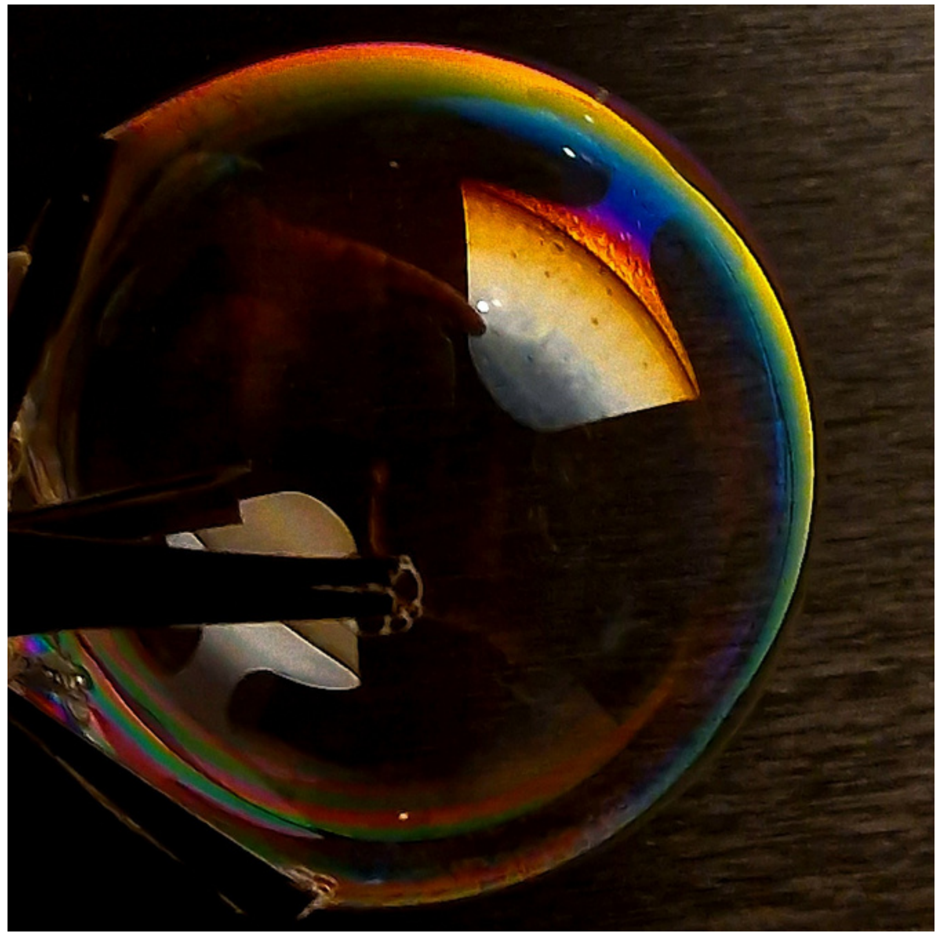
An important and current research topic is the analysis of aversion and blocks towards mathematics occurring in young children. Some causes of this troubling phenomenon are related to the teacher. These are substantive conditions (insufficient knowledge of mathematics), psycho-pedagogical (insufficient knowledge about the child, about what his perceptual capabilities are at each stage of development), and methodological (related to the teacher's ability to design the educational process well, including defining goals and choosing appropriate methods, forms, and teaching aids)[7]. However, it must be emphasized that the most important difficulty lies in overcoming by the child's distance, which separates their way of thinking (based on sensory concrete cognition) from the realm of hypothetical-deductive reasoning conducted on abstract objects. [...] The uniqueness of mathematics among other subjects lies in that its objects cannot be sensed. They are invisible, without color, mass, smell, or temperature. They are abstract, formal. In everyday life, we use physical representatives of mathematical concepts [8].

The difficulty of sensory understanding of mathematics was emphasized over a hundred years ago by the renowned Cambridge mathematician, physicist, and philosopher, Alfred North Whitehead: 'Geometry is a science as abstract as other branches of mathematics. It deals with the study of shapes and the relative positions of figures, and it does not care who perceives the given things and how they become acquainted with them: through hearing, touch, or sight. [...] Observing individual examples of cones, spheres, cylinders, etc., undoubtedly aids our imagination, but the propositions of geometry concern all figures of a certain kind, not just those we have in front of us at any given moment' [9]. In the mathematical education of young children, I encourage the use of methods based on reductive reasoning rooted in sensory representations [10]. They are an indispensable basis for heuristic inquiries that replace or supplement classic deductive strategies. Remember that children should not be given formal explanations based on logic too early, as it is completely foreign to the child's mind (until a certain age), and its implications are sterile for them [11].



Visiting a Spanish kindergarten, I noticed a board with attached papers in the shape of squares. Of different sizes and colors. Attached at various angles. Saturating with different representations of the idea of 'square' was meant to bring children closer to the essential features of this concept. Wacław Zawadowski emphasizes that mathematics begins where the child starts to differentiate a triangle (conceived) from an object in the shape of a triangle [12]. I am skeptical of the commonly used concept of teaching children to distinguish rectangles and squares as separate categories. Although the names of these figures are listed side by side in the Polish core curriculum, one should consider the risk of recognizing the correct relationships between a square and a rectangle.

The risk concerns building categories by the child and shaping correct logical reasoning. I suggest mentioning as early as possible that a rectangle, which has all sides of the same length, also has a second name: 'square'. In children's mathematical education, we often use the so-called ostensive definition. When introducing a concept, we avoid formulating a classic definition, but by giving a certain number of examples - representatives, we try to saturate the imagination about the concept. The number of examples will be sufficient when the child can detach their thinking from the color, size of the figure, and come closer to invariant properties that indicate that a given figure can be called a square or a triangle.



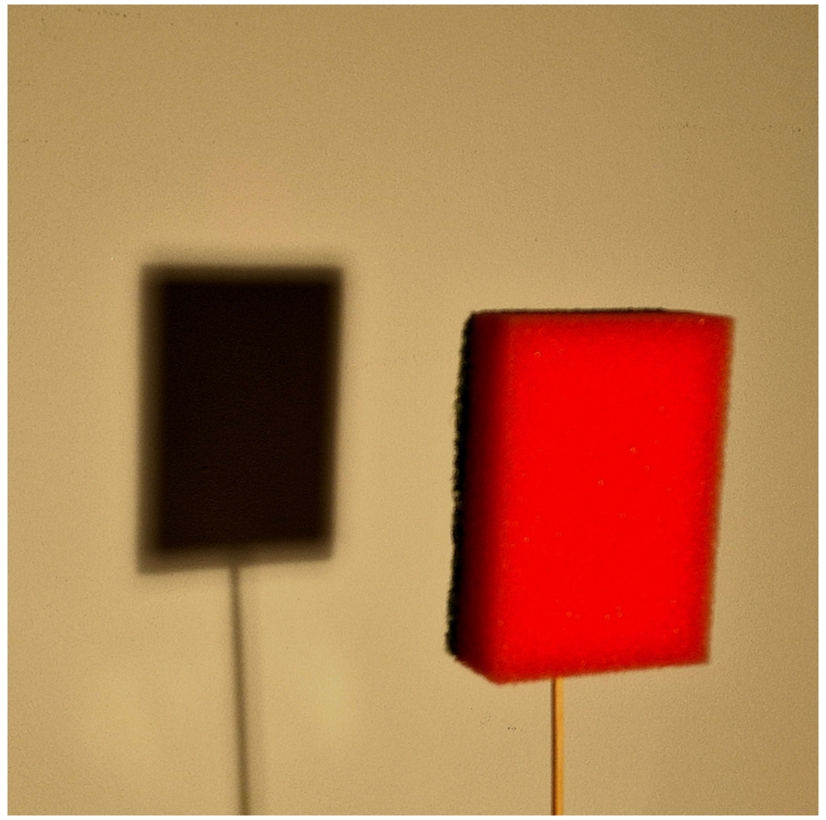
When introducing a child to the world of mathematics, nothing should be taught that would have to be referred to as untrue in the future. It’s a difficult task because, after all, telling the truth and only the truth, we cannot always present the whole... The spiral nature of the curriculum assumes repeated contemplation over the concept. This contemplation is associated with a gradual approach to the mathematical abstract. At first, this approach is 'coarse', imprecise, e.g., when learning about the concept of a sphere, first a ball or an orange peel appears, then a soap bubble, whose surface is so fleeting and thin that combined with transparency, it can be considered a quite decent representation of this concept.

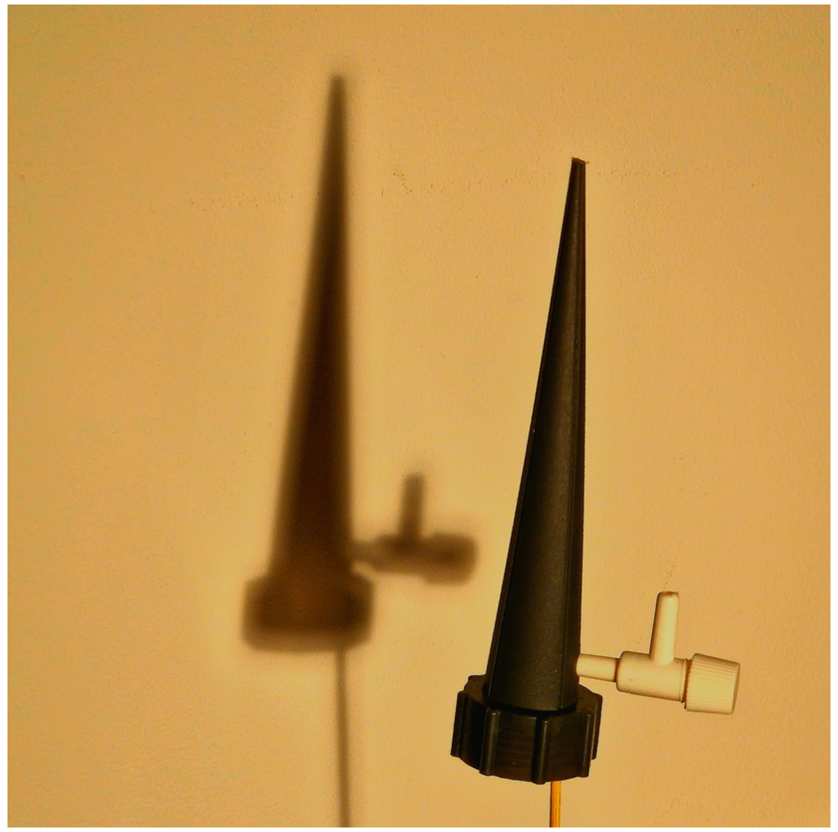
A relatively simple teaching strategy is to reverse the order of learning geometric shapes by children. I believe that it is inappropriate to start children's education with learning flat figures and then move on to spatial figures. This contradicts the principle of grading difficulty because spatial figures are simply closer to the child. After all, they encounter them already in infancy (rattles, balls). Flat figures are immaterial.

They don’t have thickness, they don’t have mass, they can’t be held. Such, seemingly reversed, strategy moving from spatial to flat figures is represented by many researchers whose scientific passion is rooted in teaching practice.
The process of abstracting flat and linear figures from the concrete of solids happens [...] naturally and then there is no need to explain to the child that a square or rectangle, forming the wall of a cube, doesn’t have the thickness that their model cut out of plywood, sheet metal, cardboard, or paper has [13]. Early geometry teaching should be associated with solids, which are closer and more imaginable to children than flat figures. Remember that the child has an innate ability to see in three dimensions. Therefore, they should be introduced from the very beginning to various figures in three-dimensional space [14].

I saw how a teacher, wanting to introduce a child to the shape of a rectangle, showed a white sheet of paper. I remember the reaction to this demonstration by a resolute second-grader who protested, 'my mom bought 500 such rectangles yesterday and that pack was heavy, it was about this thick' (he showed about 5 cm with his fingers). The child accompanied his mother during the purchase of printer paper. His observations triggered a better sense of the 'spatiality' of the sheet than the teacher's. Incidentally, in my teaching practice, I rated such remarks, questions, or comments from my students with praise and high marks. A student asked me, 'how so? A grade for a question asked by a student?'. I tried to make the future teacher understand on this example the incredibly important function of school assessment, far from verifying so-called educational effects, but consisting in supporting and encouraging the student to attention and cognitive courage, to activity and curiosity. How to introduce a child to the world of flat figures when you can’t touch these figures? Many people use Dienenes block sets for this purpose. When playing cognitively with these colorful blocks, remember the special role of the supporting adult and counteract the risk of improper formation of concepts of flat figures when naming prisms of low height. In this book, I present two methods tested by me.

The first is related to shadow theater. Playing with generating the shapes of shadows cast by spatial figures can be conducted at home, for example, in the evening before bedtime. The game does not require expensive teaching aids. A lamp, a wall, and a few objects of shapes similar to a rectangular prism, cone, or ball are enough. In this part of the book dedicated to the child, in the photographs, you can see everyday objects: a dish sponge, a paper cap, or a ball casting flat shadows on the wall with rectangular, triangular, or circular shapes. The shadow, in my opinion, is the best way to show a flat figure. Deprivation of color, as well as material character (thickness, mass), makes this representation almost ideal. When we turn off the light source, the image disappears, but it can be recreated after a moment. The interactive nature of the mathematical shadow theater allows for the selection of objects, their positioning, distance from the screen. The child, in the course of such experiments, for the first time notices the ambiguity of projection. They independently indicate models of various spatial figures casting identical shadows (e.g., a cylinder and a rectangular prism casting a rectangular shadow).In an (as yet) unconscious way, they encounter a function that is not one-to-one. The guessing game 'which figure cast a given shadow' resembles Plato's cave, where the stone wall is replaced by a screen, and the fire – by a flashlight or lamp. Over time, children's observations will crystallize towards understanding the properties of similarity, characteristics of similarity, and congruence. Projection in the kitchen or bathroom (where there are rectangular tiles) can turn out to be extremely educational if the adult guides the game properly.





Observing square shadows, at a higher level of play, may lead the preschooler to original discoveries 'for every square, you can set up a square obstacle so that the shadow fits exactly into the square tile', which in the future will result in the statement that 'all squares are similar to each other" and "there are rectangles that cannot be arranged in any way so that their shadows match the rectangular tiles".In the spiral of children's understanding of mathematics, at this point, the seeds of later school mathematics concepts such as proportion, similarity, or isometry begin to appear. Such child observations can be classified as manifestations of mini-c creativity, which James Kaufman and Ronald Beghetto recognize as a catalyst and guide to everyday and exceptional creativity [15]. Children's discoveries have a strong emotional coloring and unleash a series of creative ideas encompassing both elementary cognitive processes (fluid creativity) and ideas crystallizing in the form of striving to solve the presented problem, understanding its structure, meaning, or context (crystallized creativity) [16].

Another method I have repeatedly tested for shaping children's perceptions of flat figures is stamps. In the book, I propose a game with stamps made from vegetables or a kitchen sponge and poster paints. The natural shape of the onion cross-section encourages discussing the circle, a rectangular prism cut from a beetroot – the rectangle, and a carrot in the shape of a cylinder – the circle. This is an inexpensive and handy game that can accompany meal preparation. A kitchen sponge can be quickly and easily shaped into the desired form. All you need are scissors and an idea. With such prepared stamps, the child creates their own compositions by multiplying the imprints of individual stamps.

In both presented examples, I have tried to capture the transfer described by Jean Piaget between manipulation of real objects and mental activities. The child gradually moves from physical activity, i.e., actions on material objects, to imaginative activity, and then to logical-mathematical activity, which is expressed in the ability to perform operations, i.e., reversible mental actions [17]. Both proposals are characterized by easy and inexpensive execution. They are very popular with the youngest and encourage a range of modifications and creative activities."

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Conclusion and Introduction to the Part Dedicated to Preschoolers

The book has been prepared in the convention of open art. Its task is also to invite the Reader to their own, constantly changing readings. Adults - to arrange educational situations and to open children's eyes to mathematics. Educators - to reflect on contemporary methods of supporting formal and informal mathematical education of children. Children - to creative play that brings new knowledge and skills.

The book consists of two parts: aimed at educators, guardians, parents, and caregivers of preschool-aged children, and a part painted with light and sound, intended for the youngest readers. I tried to combine different meaningful messages so as to best adapt their content and form to a specific recipient. The different sign material of the messages seems to completely differentiate the texts (e.g., literary and photographic), but [...] it turns out that they can be found to have analogous structures, which testify to the community of cultural texts in the sphere of poetics; these structures enable messages representing different media to interact [1]. The book remains at a certain distance from classical publications, which are places of rest and immobilization of words [2].

The photographs and recordings dedicated to preschool children contained in it are intended to direct their perception and enrich the interpretative field towards mathematics. The multidirectionality and ambiguity accompanying the reading of photographs are intended to stimulate attention and encourage interpretative effort. It should provoke creative decoding of information and lead to a deeper understanding of mathematical concepts. The use of the narrator's voice and many digital photographs in the propaedeutic shaping of a child's mathematical thinking is not just an example of the organization of educational content. This is a theoretically grounded example of the intertwining of two components of our reality: real and digital, known in pedagogy as complementary education [3]. Complementarity can also be read in the constant movement between the three representations described by Jerome Bruner. Images created based on concrete real activities provide a preconceptual structure allowing for the reconstruction of cognitive classes even at the preoperational level [4].

Today's digital media, thanks to the Internet and digital recording capabilities, allow for crossing territorial and temporal barriers. They fill the space around us today. They bring with them the risk of enslavement and misinformation, but at the same time offer the chance for effective education, adapted to the needs of a 21st-century child. Sometimes, a child is better oriented in the intricacies of exploiting electronic devices than their parents. They consider them ubiquitous and obvious. Feels safe when online. Digital photography is an extremely popular way of interpersonal influence. Among other digital media, it stands out for its credibility and universality. It serves to communicate, document, capture, and share with others the feelings accompanying perception. Photography is also a multidimensional educational medium, the functions and possibilities of which I presented in my educational concept called mathematical photoeducation [6]. This concept does not focus on the photographs themselves, but on the person who makes them and who experiences them. It activates reproductive and creative thought processes, brings abstract concepts and regularities closer both through visual metaphors, as text. [7]

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Recommendations

Małgorzata Makiewicz's book is another interesting publication in the field of young readers' interest in mathematics. In a content- and visually-attractive way, the author introduces the reader to the issues and development of the skills of counting, measuring, weighing and getting to know figures through play. This is an example of a creative and innovative proposal for the education of a preschool child in the field of mathematics. The child, together with parents, teacher or other caregiver, learns about mathematical content and sees it in his/her life. It is a background to prepare the child for school education and to remove the fear of mathematics or the disbelief in his/her abilities, mathematical skills. The presented and described creative visual-mathematical games shape the child's person, emotions, thinking, and creative approach to finding solutions in mathematical play. The publication will inspire both parents and teachers to have cognitively interesting mathematical games with children and thus mathematics.

Professor Urszula Szuścik

The book: 'Math&Art. Look, it's mathematics' is in the line of Professor Małgorzata Makiewicz's search for the golden key opening children's and young people's hearts and minds to mathematical knowledge and skills. Striving for the development of mathematical competences is accompanied not only by the author's substantive knowledge, but also exceptional care for the linguistic and visual aesthetics of the work, efficient use of multimedia tools and sound knowledge of human cognitive development.
It is therefore no surprise that Professor Małgorzata Makiewicz's work aimed at the development of mathematical culture has been generating so much scientific and, more broadly, social interest for years. This has been the case both with the world's first international photographic competition 'Mathematics in Lens', created by Professor Makiewicz, and with the proposals submitted to outstanding publications popularizing mathematical culture. They arouse the interest not only of academics and people fascinated by mathematics, but also - and this is important - of teachers and parents, who understand the importance of developing personal competences in this field from the very beginning of childhood.
The book: 'Math&Art. Look, it's mathematics' proves that Professor Małgorzata Makiewicz has discovered the golden key which opens unknown paths into the mysterious world of mathematical adventure. The intellectual journey proposed by the Author moves minds, teaches to look and understand. It is a source of extraordinary experiences and mathematical fascinations.
It is also noteworthy that the Author reveals a unique talent in using tools that appeal to many senses of the reader, viewer and listener. This polysensoriality of the work constitutes its immense value. This is not only because of its originality, but above all because it appeals to the tastes of people living in the intermingling spaces of the real and virtual worlds, living in a period of rapid development of artificial intelligence, biotechnology, robotics and quantum computers.

Professor Maciej Tanaś